ELSEVIER Journal of Geometry and Physics 27 (1998) 281–296

JOURNAL OF GEOMETRY AND PHYSICS

Coeffective and de Rham cohomologies of symplectic manifolds

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Received 26 June 1997

Abstract

We discuss the relation of the coeffective cohomology of a symplectic manifold with the topology of the manifold. A bound for the coeffective numbers is obtained. The lower bound is got for compact Kähler manifolds, and the upper one for non-compact exact symplectic manifolds. A Nomizu's type theorem for the coeffective cohomology is proved. Finally, the behaviour of the coeffective cohomology under deformations is studied. © 1998 Elsevier Science B.V.

Subj. Class.: Differential geometry
1991 MSC: 53C15
Keywords: Symplectic manifolds; Kähler manifolds; Coeffective cohomology; Deformations

1. Introduction

The coeffective cohomology of a symplectic manifold (M, ω) was introduced by Bouché [4] as the cohomology of the coeffective subcomplex $(\mathcal{A}^*(M), d)$ of the de Rham complex of M, where $\mathcal{A}^k(M) = \{\alpha \in \Lambda^k(M) \mid \alpha \wedge \omega = 0\}$. A natural question is to look for a relation between the coeffective and de Rham cohomologies of the symplectic manifold. Bouché

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PII S0393-0440(97)00084-3

has proved that the coeffective complex is elliptic, and hence its cohomology groups have finite dimension for compact symplectic manifolds. Moreover, he proved that for a compact Kähler $2n$-dimensional manifold there is the isomorphism

$$H^k(\mathcal{A}(M)) \cong \widetilde{H}^k(M) \quad \forall k \neq n, \tag{1}$$

where $H^k(\mathcal{A}(M))$ denotes the coeffective cohomology group of degree k and $\widetilde{H}^k(M)$ is the subspace of the de Rham cohomology group $H^k(M)$ consisting of those classes $a \in H^k(M)$ such that $a \wedge [\omega] = 0$, or in other words, the truncated de Rham cohomology group of degree k. Notice that for an arbitrary symplectic manifold (compact or not) we have $H^k(\mathcal{A}(M)) = 0$ for $k \leq n-1$, where $\dim M = 2n$.

In [1] the authors exhibited a compact symplectic manifold R^6 for which the above isomorphism does not hold. Moreover, R^6 does not admit any Kähler structure. The harder task to obtain this result was the computation of the coeffective cohomology. In fact, for a compact nilmanifold or completely solvable manifold Γ/G, Nomizu's theorem permits us to calculate in a very simple way the de Rham cohomology in terms of the cohomology of the Lie algebra of the Lie group G. A similar result for the coeffective cohomology was obtained in [10]. This last result has permitted to exhibit a large family of examples [8–10]. Using a technique based on the long exact sequence in cohomology associated with an exact short sequence of complexes, we obtain in the present paper a very simple proof of Nomizu's theorem for the coeffective cohomology.

The aim of the present paper is to discuss the relation of the coeffective cohomology with the topology of the symplectic manifold.

First of all, and using a technique based on the long exact sequence in cohomology associated with an exact short sequence of complexes, we obtain that the coeffective cohomology groups of a symplectic manifold (M, ω) of finite type have finite dimension (called the coeffective numbers), and moreover, they satisfy the following inequalities:

$$b_k(M) - b_{k+2}(M) \leq c_k(M) \leq b_k(M) + b_{k+1}(M).$$

As a consequence, for a compact Kähler manifold, we deduce that

$$c_k(M) = b_k(M) - b_{k+2}(M), \quad k \geq n+1,$$

which means that the coeffective numbers of a compact Kähler manifold measure the jumps between the Betti numbers. The situation is dramatically different for non-compact symplectic manifolds. In fact, we prove that, if (M, ω) is an exact symplectic $2n$-dimensional manifold, we have

$$c_k(M) = b_k(M) + b_{k+1}(M) \quad \text{for } k \geq n+1.$$

Using the above formula, we construct a non-compact Kähler manifold for which (1) is not satisfied.

We also discuss the behaviour of the coeffective cohomology when the symplectic structure is deformed. More precisely, we prove that the coeffective cohomology of a symplectic manifold is invariant by isotopies, but not by pseudo-isotopies.

2. Definitions and basic facts

Let M be a real $2n$-dimensional smooth manifold, $\mathfrak{X}(M)$ the Lie algebra of vector fields on M and $\Lambda^k(M)$ be the space of k-forms on M. A symplectic structure on M is a 2-form $\omega \in \Lambda^2(M)$ closed (that is, $\mathrm{d}\omega = 0$) and non-degenerate (that is, $\omega^n \neq 0$). The pair (M, ω) is called a *symplectic manifold.*

On the other hand, an almost Hermitian structure on a $2n$-dimensional manifold M is a pair (g, J) of a Riemannian metric g and an almost complex tensor J such that g and J are compatibles:

$$g(JX, JY) = g(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Then it can be defined as the associated fundamental 2-form (also called the Kähler form):

$$\omega(X, Y) = g(JX, Y), \quad X, Y \in \mathfrak{X}(M).$$

It is easy to see that $\omega^n \neq 0$. Furthermore, if $\mathrm{d}\omega = 0$, then ω defines a symplectic structure (in this case, (g, J) is said to be an almost Kähler structure). Conversely, it is known [5,19,24] that given a symplectic structure ω on M, there exists an almost Hermitian structure (g, J) such that ω is its fundamental 2-form (notice that the almost Hermitian structure is not unique).

A $2n$-dimensional manifold is said to be a *Kähler manifold* if there exists an almost Hermitian structure (g, J) such that the fundamental 2-form ω is closed and J is integrable (that is, J defines a complex structure on M) (for a more detailed study see [15–17]).

Let (M, ω) be a $2n$-dimensional symplectic manifold. If d denotes the exterior derivative on M, then we have the de Rham differential complex

$$\cdots \longrightarrow \Lambda^{k-1}(M) \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \longrightarrow \cdots,$$

whose cohomology $H^*(M)$ is the de Rham cohomology of M. Also, let

$$\mathcal{A}^k(M) = \{\alpha \in \Lambda^k(M) \mid \alpha \wedge \omega = 0\}$$

be the subspace of $\Lambda^k(M)$ of coeffective forms on M (in the case that we are considering more than one structure on the manifold we shall add a reference to the structure on the notation). Alternatively, we can introduce the linear mapping $L : \Lambda^k(M) \longrightarrow \Lambda^{k+2}(M)$ defined by $L(\alpha) = \alpha \wedge \omega$ and hence $\mathcal{A}^k(M) = \mathrm{Ker}\,\{L : \Lambda^k(M) \longrightarrow \Lambda^{k+2}(M)\}$. Since ω is closed, L and d commute, which implies that

$$\cdots \longrightarrow \mathcal{A}^{k-1}(M) \xrightarrow{d} \mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M) \longrightarrow \cdots$$

is a differential subcomplex of the de Rham complex. Its cohomology $H^k(\mathcal{A}(M))$ is called *coeffective cohomology* of the symplectic manifold M.

Proposition 2.1 [17]. *Let M be a $2n$-dimensional symplectic manifold. Then L is injective for $k \leq n-1$ and surjective for $k \geq n-1$.*

Corollary 2.1. $\mathcal{A}^k(M) = \{0\}$ *for $k \leq n-1$; and as a consequence,* $H^k(\mathcal{A}(M)) = \{0\}$ *for $k \leq n-1$.*

On the other hand, since the fundamental 2-form ω is closed, it defines a de Rham cohomology class $[\omega] \in H^2(M)$. Then we can consider the de Rham cohomology groups truncated by the class of the fundamental 2-form ω, that is,

$$\widetilde{H}^k(M) = \{a \in H^k(M) \mid a \wedge [\omega] = 0\}.$$

The relation between the coeffective cohomology and the de Rham cohomology truncated by $[\omega]$ has been discussed in [4]. In fact, for a compact Kähler manifold M Bouché has proved that

$$H^k(\mathcal{A}(M)) \cong \widetilde{H}^k(M) \quad \forall k \neq n, \tag{2}$$

where dim $M = 2n$. The result does not hold for arbitrary symplectic manifolds as we have proved in [1] (see also [10]).

Remark 2.1. A coeffective cohomology can be defined on an arbitrary manifold endowed with a closed 2-form. For instance, for an almost cosymplectic $(2n+1)$-dimensional manifold M with almost cosymplectic structure (η, Ω) (that is, η is a closed 1-form and Ω is a closed 2-form such that $\eta \wedge \Omega^n$ is a volume form on M), we can consider the coeffective complex determined by Ω. This cohomology was introduced by Chinea et al. [6]. It has been also studied in [1,10].

3. Exact sequences and coeffective cohomology

The aim of this section is to relate the coeffective cohomology with the de Rham cohomology by means of a long exact sequence in cohomology. As in Section 2, let M be a symplectic manifold of dimension $2n$ with symplectic form ω. Then, taking into account the mapping $L : \Lambda^k(M) \longrightarrow \Lambda^{k+2}(M)$, we consider the following natural short exact sequence for any degree k:

$$0 \longrightarrow \operatorname{Ker} L = \mathcal{A}^k(M) \stackrel{i}{\longrightarrow} \Lambda^k(M) \stackrel{L}{\longrightarrow} \operatorname{Im}^{k+2} L \longrightarrow 0. \tag{3}$$

Since L and d commute, then (3) becomes a short exact sequence of differential complexes:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \longrightarrow & & \mathrm{Im}^{k+1} L & \xrightarrow{d} & \mathrm{Im}^{k+2} L & \xrightarrow{d} & \mathrm{Im}^{k+3} L & \longrightarrow \cdots & \\
 & & \uparrow L & & \uparrow L & & \uparrow L & & \\
\cdots \longrightarrow & & \Lambda^{k-1}(M) & \xrightarrow{d} & \Lambda^{k}(M) & \xrightarrow{d} & \Lambda^{k+1}(M) & \longrightarrow \cdots & \\
 & & \uparrow i & & \uparrow i & & \uparrow i & & \\
\cdots \longrightarrow & & \mathcal{A}^{k-1}(M) & \xrightarrow{d} & \mathcal{A}^{k}(M) & \xrightarrow{d} & \mathcal{A}^{k+1}(M) & \longrightarrow \cdots & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Therefore, we can consider the associated long exact sequence in cohomology [12]:

$$\cdots \longrightarrow H^k(\mathcal{A}(M)) \xrightarrow{H(i)} H^k(M) \xrightarrow{H(L)} H^{k+2}(\mathrm{Im}\, L) \xrightarrow{C_{k+2}} H^{k+1}(\mathcal{A}(M)) \longrightarrow \cdots, \tag{4}$$

where $H(i)$ and $H(L)$ are the induced homomorphisms in cohomology by i and L, respectively, and C_{k+2} is the connecting homomorphism defined in the following way: for $[\alpha] \in H^{k+2}(\mathrm{Im}\, L)$, then $C_{k+2}[\alpha] = [\mathrm{d}\beta]$ for $\beta \in \Lambda^k(M)$ such that $L\beta = \alpha$.

From Proposition 2.1 we know that $\mathrm{Im}^{k+2} L = \Lambda^{k+2}(M)$ for $k \geq n-1$. As a consequence,

$$H^{k+2}(\mathrm{Im}\, L) = H^{k+2}(M)$$

for $k \geq n$. Furthermore, the long exact sequence in cohomology (4) may be expressed as

$$\cdots \longrightarrow H^k(\mathcal{A}(M)) \xrightarrow{H(i)} H^k(M) \xrightarrow{H(L)} H^{k+2}(M) \xrightarrow{C_{k+2}} H^{k+1}(\mathcal{A}(M)) \longrightarrow \cdots \tag{5}$$

for such degrees. Now, we shall decompose the long exact sequence (5) in 5-term exact sequences:

$$\begin{aligned} 0 \to\ & \mathrm{Ker}\, H(i) \\ &= \mathrm{Im}\, C_{k+1} \xrightarrow{i} H^k(\mathcal{A}(M)) \xrightarrow{H(i)} H^k(M) \xrightarrow{H(L)} H^{k+2}(M) \xrightarrow{C_{k+2}} \mathrm{Im}\, C_{k+2} \to 0. \end{aligned} \tag{6}$$

If M is of finite type, the de Rham cohomology groups have finite dimension. Denote by $b_k(M) = \dim H^k(M)$ the Betti numbers of M. Since $0 \leq \dim(\mathrm{Im}\, C_k) \leq b_k(M)$ for $k \geq n+2$ we have the following result.

Proposition 3.1. *Let M be a symplectic $2n$-dimensional manifold of finite type. Then the coeffective cohomology group $H^k(\mathcal{A}(M))$ has finite dimension for $k \geq n+1$. Thus, we define the coeffective numbers $c_k(M) = \dim H^k(\mathcal{A}(M))$ for $k \neq n$. (Remember that $c_k = 0$ for $k \leq n-1$.)*

Remark 3.1. Notice that because the de Rham cohomology groups of a compact manifold have finite dimension, we immediately deduce from Proposition 3.1 that the coeffective cohomology groups of a compact symplectic $2n$-dimensional manifold have also finite dimension for $k \neq n$ [4].

From (6), we have

$$\dim(\operatorname{Im} C_{k+1}) - \dim H^k(\mathcal{A}(M)) + \dim H^k(M) - \dim H^{k+2}(M) + \dim(\operatorname{Im} C_{k+2}) = 0$$

for $k \geq n+1$, from which we deduce

$$\dim(\operatorname{Im} C_{k+1}) - c_k(M) + b_k(M) - b_{k+2}(M) + \dim(\operatorname{Im} C_{k+2}) = 0. \tag{7}$$

Now, we deduce that the coeffective numbers are bounded by upper and lower limits depending on the Betti numbers of the manifold.

Theorem 3.1. *For $k \geq n+1$, we have*

$$b_k(M) - b_{k+2}(M) \leq c_k(M) \leq b_k(M) + b_{k+1}(M). \tag{8}$$

Now, we shall see the behaviour of some examples of compact symplectic manifolds with respect to inequalities (8). The calculation of the coeffective numbers for these examples is possible, thanks to a Nomizu's type theorem for the coeffective cohomology in the cases of compact nilmanifolds and completely solvable manifolds given in [10] (see also Section 6).

Example 3.1 [1,10]. Consider the six-dimensional compact nilmanifold $R^6 = \Gamma\backslash G$, where G is a simply connected nilpotent Lie group of dimension 6 defined by the left invariant 1-forms $\{\alpha_i,\ 1 \leq i \leq 6\}$ such that

$$\mathrm{d}\alpha_i = 0, \quad 1 \leq i \leq 3, \qquad \mathrm{d}\alpha_4 = -\alpha_1 \wedge \alpha_2,$$
$$\mathrm{d}\alpha_5 = -\alpha_1 \wedge \alpha_3, \qquad \mathrm{d}\alpha_6 = -\alpha_1 \wedge \alpha_4,$$

and Γ is a discrete and uniform subgroup of G.

It will be convenient to introduce an abbreviated notation for wedge products; we write $\alpha_{ij} = \alpha_i \wedge \alpha_j$, $\alpha_{ijk} = \alpha_i \wedge \alpha_j \wedge \alpha_k$, and so forth.

An easy computation, using Nomizu's theorem (see Section 6), shows that the de Rham cohomology of R^6 is:

$$H^0(R^6) = \{1\},$$
$$H^1(R^6) = \{[\alpha_1], [\alpha_2], [\alpha_3]\},$$

$$H^2(R^6) = \{[\alpha_{15}], [\alpha_{16}], [\alpha_{23}], [\alpha_{24}], [\alpha_{35}], [\alpha_{25} + \alpha_{34}]\},$$
$$H^3(R^6) = \{[\alpha_{135}], [\alpha_{145}], [\alpha_{146}], [\alpha_{156}], [\alpha_{234}], [\alpha_{235}], [\alpha_{246}], [\alpha_{236} + \alpha_{245}]\},$$
$$H^4(R^6) = \{[\alpha_{1246}], [\alpha_{1256}], [\alpha_{1356}], [\alpha_{1456}], [\alpha_{2345}], [\alpha_{2346}]\},$$
$$H^5(R^6) = \{[\alpha_{12456}], [\alpha_{13456}], [\alpha_{23456}]\},$$
$$H^6(R^6) = \{[\alpha_{123456}]\}.$$

Thus, the first Betti number of R^6 is $b_1(R^6) = 3$, and hence R^6 does not admit Kähler structures. However, R^6 admits symplectic structures. For instance,

$$\omega = \alpha_{15} + \alpha_{16} + \alpha_{25} + \alpha_{34} + \alpha_{13}$$

is a symplectic form on R^6. Next, using Theorem 6.2, we get

$$b_4(R^6) - b_6(R^6) = 5 < c_4(R^6) = 6 < b_4(R^6) + b_5(R^6) = 9,$$
$$b_5(R^6) = c_5(R^6) = 3 < b_5(R^6) + b_6(R^6) = 4.$$

Example 3.2 [10]. Consider the eight-dimensional compact solvmanifold $M^8 = \Gamma/G$, where G is a simply connected completely solvable Lie group of dimension 8 defined by the left invariant 1-forms $\{\alpha_i,\ 1 \le i \le 8\}$ such that

$$\mathrm{d}\alpha_i = 0, \quad 1 \le i \le 3,$$
$$\mathrm{d}\alpha_4 = -\alpha_1 \wedge \alpha_2, \qquad \mathrm{d}\alpha_5 = -\alpha_1 \wedge \alpha_3, \qquad \mathrm{d}\alpha_6 = -\alpha_1 \wedge \alpha_4,$$
$$\mathrm{d}\alpha_7 = -\alpha_1 \wedge \alpha_7, \qquad \mathrm{d}\alpha_8 = \alpha_1 \wedge \alpha_8,$$

and Γ is a uniform subgroup of G.

As in the above example, we introduce an abbreviated notation for wedge products. The manifold M^8 does not admit Kähler structures since $b_1(M^8) = 3$, but it has symplectic structures. Indeed, the 2-form given by

$$\omega = \alpha_{15} + \alpha_{16} + \alpha_{25} + \alpha_{34} + \alpha_{78}$$

is symplectic. By using Nomizu's theorem and Theorem 6.2, we have

$$b_5(M^8) - b_7(M^8) = 8 = c_5(M^8) < b_5(M^8) + b_6(M^8) = 18,$$
$$b_6(M^8) - b_8(M^8) = 6 = c_6(M^8) < b_6(M^8) + b_7(M^8) = 10,$$
$$b_7(M^8) = 3 = c_7(M^8) < b_7(M^8) + b_8(M^8) = 4.$$

4. Compact symplectic manifolds

In this section, we shall use the long exact sequence in cohomology (5) and the Hodge theorem [25] to obtain some results on the coeffective cohomology for compact manifolds. In particular, we shall prove that the coeffective cohomology is a topological invariant for compact Kähler manifolds.

Theorem 4.1. *Let M be a compact Kähler manifold of dimension $2n$. Then we have*

$$c_k(M) = b_k(M) - b_{k+2}(M), \quad k \geq n+1. \tag{9}$$

Proof. To prove this result it is sufficient to show that the mapping C_{k+2} identically vanishes, then from (7) we obtain relation (9).

Let $a \in H^{k+2}(M)$. Let α be the unique harmonic representative of the de Rham cohomology class a. Since the map $L : \Lambda^k_H(M) \longrightarrow \Lambda^{k+2}_H(M)$ (where $\Lambda^k_H(M)$ is the space of harmonic k-forms) is surjective for $k \geq n-1$ [4], then there exists a harmonic k-form β such that $L\beta = \alpha$. From the definition of the connecting homomorphism, $C_{k+2}[\alpha] = [\mathrm{d}\beta] = 0$. □

Remark 4.1. Since the Betti numbers are topological invariants for compact manifolds [25], Theorem 4.1 implies that the coeffective cohomology groups are topological invariants for compact Kähler manifolds. Moreover, since

$$b_k(M) \geq b_{k+2}(M), \quad k \geq n,$$

for compact Kähler manifolds, we have proved that the coeffective numbers measure these jumps on the Betti numbers.

For a compact Kähler manifold we know that $H^k(\mathcal{A}(M)) \cong \widetilde{H}^k(M)$ for $k \neq n$ [4] but this is not true in general for arbitrary compact symplectic manifolds [1,10]. For instance, Examples 3.1 and 3.2 are compact symplectic manifolds which do not satisfy the isomorphism (2).

5. Non-compact symplectic manifolds

In differential geometry and physics there exist important examples of non-compact symplectic manifolds. The main example is $\mathbb{R}^{2n}$ with the standard structure, but another interesting example is the contangent bundle of a manifold with the canonical exact symplectic structure [16,17]. Exact symplectic manifolds are those that occur most commonly in mechanical problems and in other physical applications. This section is devoted to the study of the coeffective cohomology in the non-compact case. In particular, we shall show which of the properties satisfied by the coeffective cohomology for compact Kähler manifolds still hold in the non-compact case.

The first example of a non-compact symplectic manifold is $\mathbb{R}^{2n}$ with the standard symplectic structure ω_0, that is,

$$\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}x_{n+1} + \cdots + \mathrm{d}x_n \wedge \mathrm{d}x_{2n},$$

where $(x_1, \ldots, x_{2n})$ are the natural coordinates on $\mathbb{R}^{2n}$. The Betti numbers of $\mathbb{R}^{2n}$ are well known:

$$b_0(\mathbb{R}^{2n}) = 1, \qquad b_k(\mathbb{R}^{2n}) = 0 \quad \text{for } k \geq 1. \tag{10}$$

On the other hand, from a direct computation (see [4]), we obtain the coeffective numbers for the standard symplectic structure ω_0 on $\mathbb{R}^{2n}$:

$$c_k(\mathbb{R}^{2n}) = 0 \quad \text{for } k \neq n.$$

More than this, we have the following.

Proposition 5.1. *Let ω be any symplectic structure on $\mathbb{R}^{2n}$ (notice that there exist exotic symplectic structures on $\mathbb{R}^{2n}$, that is, no standard symplectic structures* [2,13,19]*) or on any smooth manifold M homeomorphic to $\mathbb{R}^{2n}$. Then*

$$c_k(M) = c_k(\mathbb{R}^{2n}) = 0 \quad \text{for } k \neq n.$$

Also, the isomorphism (2) *between the coeffective cohomology and the de Rham cohomology truncated by $[\omega]$ is satisfied for $k \neq n$.*

Proof. The result for $\mathbb{R}^{2n}$ is obtained directly from (10) and (5). Moreover, if M is homeomorphic to $\mathbb{R}^{2n}$, then $b_k(M) = b_k(\mathbb{R}^{2n})$ and the result follows as for $\mathbb{R}^{2n}$. □

Some partial results of the above situation are given in the following theorems.

Theorem 5.1 [18]. *The Kähler form ω on a simply connected complete Kähler $2n$-dimensional manifold M of non-positive sectional curvature is diffeomorphic to the standard symplectic form ω_0 on $\mathbb{R}^{2n}$. This means in particular that the symplectic structure on a Hermitian symmetric space of non-compact type is standard.*

Theorem 5.2 (McDuff, Floer, Eliashberg [5]). *Suppose that a $2n$-dimensional manifold M is asymptotically flat and contains no symplectic spheres. Then M is diffeomorphic to $\mathbb{R}^{2n}$.*

Let us recall [5] that a non-compact symplectic manifold M of dimension $2n$ is *asymptotically flat* if there is a compact set $K_1 \subset M^{2n}$ and a compact set $K_2 \subset \mathbb{R}^{2n}$ so that $M \backslash K_1$ is symplectomorphic to $\mathbb{R}^{2n} \backslash K_2$ (with the standard symplectic structure).

Now, if we consider a symplectic structure on a non-compact complete Riemannian manifold of positive sectional curvature, then we also are in the conditions of Proposition 5.1, as it is shown in the following theorem.

Theorem 5.3 [11]. *Every complete Riemannian manifold of positive sectional curvature that is non-compact is diffeomorphic to the Euclidean space.*

A particular class of non-compact symplectic manifolds is the class of the exact symplectic manifolds, that is, those for which the symplectic form ω is exact. Note that a compact symplectic manifold is never exact.

We first exhibit an example of exact symplectic manifold which satisfies (2).

Example 5.1 (*The affine group $GA(\mathbb{R}^n)$*). Let $GA(\mathbb{R}^n) = \mathbb{R}^n \times GL(\mathbb{R}^n)$ be the group of affine transformations of $\mathbb{R}^n$. It is known [3] that the affine group $GA(\mathbb{R}^n)$ admits invariant symplectic structures, and since $H^2(GA(\mathbb{R}^n)) = 0$ all of them are exact. Moreover, we have [12]:

$$b_k(GA(\mathbb{R}^n)) = 0, \quad \forall k \geq \tfrac{1}{2}(n^2+n),$$

where n^2+n is the dimension of $GA(\mathbb{R}^n)$. Therefore, we deduce that

$$c_k(GA(\mathbb{R}^n)) = 0, \quad \forall k \neq \tfrac{1}{2}(n^2+n),$$

and for any symplectic structure on $GA(\mathbb{R}^n)$. Moreover, the isomorphism (2) is satisfied for $k \neq \frac{1}{2}(n^2+n)$. It should be noticed that the affine group $GA(\mathbb{R}^n)$ is not homeomorphic to $\mathbb{R}^{2n}$ because it has non-zero finite Betti numbers.

Theorem 5.4. *Let M be a (non-compact) exact symplectic manifold of dimension $2n$ which is of finite type. Then*

$$c_k(M) = b_k(M) + b_{k+1}(M) \quad \text{for } k \geq n+1.$$

Proof. It is sufficient to show that the mapping $H(L)$ identically vanishes, and then the result follows from (5). Assume that $\omega = -\mathrm{d}\lambda$ for some 1-form λ. Therefore, if $[\alpha] \in H^k(M)$, we have

$$H(L)[\alpha] = [L\alpha] = (-1)^{k+1}[\mathrm{d}(\alpha \wedge \lambda)] = 0. \qquad \square$$

Remark 5.1. It should be noticed that if M is a non-compact exact symplectic manifold, then, by using a similar argument to that in the proof of Theorem 5.4, we conclude that

$$\widetilde{H}^k(M) = H^k(M) \quad \forall k.$$

Corollary 5.1. *Let M be a (non-compact) exact symplectic manifold of dimension $2n$, which is of finite type and with $b_k(M) \neq 0$ for some k such that $n+2 \leq k \leq 2n$. Then*

$$H^{k-1}(\mathcal{A}(M)) \ncong \widetilde{H}^{k-1}(M) = H^{k-1}(M).$$

Suppose that M is an exact Kähler manifold satisfying the conditions of Corollary 5.1. Then the isomorphism (2) between the coeffective cohomology and the de Rham cohomology truncated by the class of the fundamental 2-form is not satisfied, in contrast with the compact case.

Now, we shall show some examples of exact symplectic manifolds.

Example 5.2 (*The symplectification of a contact manifold*). Let M be a compact contact manifold of dimension $2n+1$ with contact structure η. The symplectification of the contact manifold M is the following exact symplectic manifold of dimension $2(n+1)$, $\overline{M} = M \times \mathbb{R}$, and the symplectic structure on $\overline{M}$ is

$$\omega = \mathrm{d}(\mathrm{e}^t \eta) = \mathrm{e}^t\, \mathrm{d}t \wedge \eta + \mathrm{e}^t\, \mathrm{d}\eta.$$

Taking into account the Künneth formula we have that

$$b_k(\overline{M}) = b_k(M).$$

Moreover, if $b_k(M) \neq 0$ for some k such that $n+3 \leq k \leq 2n$, then $\overline{M}$ satisfies the conditions of Corollary 5.1, and then the isomorphism (2) is not satisfied. Let us see a particular example of this situation.

If the contact manifold M is a Sasakian manifold, then its symplectification $\overline{M}$ is a non-compact exact Kähler manifold [22]. For instance, let $M(r,1) = \Gamma\backslash G$ be the $(2r+1)$-dimensional compact nilmanifold, where G is a simply connected nilpotent Lie group of dimension $(2r+1)$ defined by the left invariant 1-forms $\{\alpha_i, \beta_i, \gamma \mid 1 \leq i \leq r\}$ such that

$$\mathrm{d}\alpha_i = \mathrm{d}\beta_i = 0, \qquad \mathrm{d}\gamma = -\sum_{i=1}^{r} \alpha_i \wedge \beta_i,$$

and Γ is a discrete and uniform subgroup of G. Then a contact structure on $M(r,1)$ is given by the contact form $\eta = \gamma$. In [7] it has been proved that this structure is Sasakian. Moreover, $b_k(M(r,1)) \neq 0$ for any $0 \leq k \leq 2r+1$.

Therefore, we have the following result.

Theorem 5.5. *For the non-compact exact Kähler manifold* $\overline{M(r,1)}$, *the isomorphism* (2) *between the coeffective cohomology and the de Rham cohomology* (*truncated by the class of the symplectic form*) *is not satisfied.*

6. Coeffective cohomology, Lie groups and homogeneous spaces

The main problem to construct examples of compact symplectic manifolds not satisfying the isomorphism (2) is the difficulty to compute the coeffective cohomology. In [10] we have proved a Nomizu's type theorem for compact symplectic nilmanifolds and completely solvable manifolds. In this section we shall prove such result by a more simple method using the long exact sequence in cohomology (5).

Let $M = \Gamma\backslash G$ be a compact nilmanifold, that is, G is a connected nilpotent Lie group with discrete subgroup Γ such that the space of right cosets $\Gamma\backslash G$ is compact. Let

$$\cdots \longrightarrow \Lambda^{k-1}(\mathfrak{g}^*) \xrightarrow{d} \Lambda^{k}(\mathfrak{g}^*) \xrightarrow{d} \Lambda^{k+1}(\mathfrak{g}^*) \longrightarrow \cdots$$

be the differential complex where $\Lambda^k(\mathfrak{g}^*)$ denotes the space of left invariant k-forms on G. Its cohomology $H^*(\mathfrak{g}^*)$ is the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$ of G. We have [21]:

Theorem 6.1 (Nomizu [21]). *Let* $M = \Gamma\backslash G$ *be a compact nilmanifold. Then there exists an isomorphism of cohomology groups*

$$H^k(\mathfrak{g}^*) \cong H^k(M).$$

(Notice that the natural map $m_{dR} : H^k(\mathfrak{g}^*) \longrightarrow H^k(M)$ *defined by* $m_{dR}[\alpha^*] = [\alpha]$, *where* $\alpha^* \in \Lambda^k(\mathfrak{g}^*)$ *and* α *is the projected k-form on* M, *is a linear isomorphism.)*

Now, let ω be a symplectic structure on M that comes from a left invariant symplectic form ω^* on G. Consider now the differential subcomplex of coeffective left invariant forms

$$\cdots \longrightarrow \mathcal{A}^{k-1}(\mathfrak{g}^*) \xrightarrow{d} \mathcal{A}^k(\mathfrak{g}^*) \xrightarrow{d} \mathcal{A}^{k+1}(\mathfrak{g}^*) \longrightarrow \cdots,$$

whose cohomology is denoted by $H^*(\mathcal{A}(\mathfrak{g}^*))$.

By similar arguments that those used in Section 3, if we define the mapping $L^* : \Lambda^k(\mathfrak{g}^*) \longrightarrow \Lambda^{k+2}(\mathfrak{g}^*)$ by $L\alpha^* = \alpha^* \wedge \omega^*$, then for any degree we get the short exact sequence

$$0 \longrightarrow \operatorname{Ker} L^* = \mathcal{A}^k(\mathfrak{g}^*) \xrightarrow{i^*} \Lambda^k(\mathfrak{g}^*) \xrightarrow{L^*} \operatorname{Im}^{k+2} L^* \longrightarrow 0$$

and, since L^* and d commute, the associated long exact sequence on cohomology similar to (4) but for left invariant forms. As L^* is surjective in the same degrees as L (see [10]), then for $k \geq n$ we deduce that

$$\cdots \longrightarrow H^k(\mathcal{A}(\mathfrak{g}^*)) \xrightarrow{H(i^*)} H^k(\mathfrak{g}^*) \xrightarrow{H(L^*)} H^{k+2}(\mathfrak{g}^*) \xrightarrow{C^*_{k+2}} H^{k+1}(\mathcal{A}(\mathfrak{g}^*)) \longrightarrow \cdots \tag{11}$$

Theorem 6.2 [10]. *Let G be a connected nilpotent Lie group endowed with a left invariant symplectic structure* ω^* *and with a discrete subgroup* Γ *such that the space of right cosets* $M = \Gamma \backslash G$ *is compact. Then the natural mapping* $m_c : H^k(\mathcal{A}(\mathfrak{g}^*)) \longrightarrow H^k(\mathcal{A}(M))$, *defined by* $m_c\{\alpha^*\} = \{\alpha\}$, *is an isomorphism of cohomology groups.*

Proof. Consider the long exact sequences in cohomology (5) and (11), and the mappings m_{dR}, m_c, that is,

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^k(\mathcal{A}(M)) & \xrightarrow{H(i)} & H^k(M) & \xrightarrow{H(L)} & H^{k+2}(M) & \xrightarrow{C_{k+2}} & H^{k+1}(\mathcal{A}(M)) & \longrightarrow \cdots \\
 & \uparrow m_c & & \uparrow m_{dR} & & \uparrow m_{dR} & & \uparrow m_c & \\
\cdots \longrightarrow & H^k(\mathcal{A}(\mathfrak{g}^*)) & \xrightarrow{H(i^*)} & H^k(\mathfrak{g}^*) & \xrightarrow{H(L^*)} & H^{k+2}(\mathfrak{g}^*) & \xrightarrow{C^*_{k+2}} & H^{k+1}(\mathcal{A}(\mathfrak{g}^*)) & \longrightarrow \cdots
\end{array}$$

for $k \geq n+1$. Since all the diagrams commute and the mappings m_{dR} are linear isomorphisms from Nomizu's theorem (Theorem 6.1), then m_c are also linear isomorphisms for $k \geq n+1$. □

(Notice that we have denoted by $[\cdot]$ the de Rham cohomology classes and by $\{\cdot\}$ the coeffective cohomology classes.)

Remark 6.1. It should be noted that for $k = n$ we obtain an injective homomorphism $H^n(\mathcal{A}(\mathfrak{g}^*)) \longrightarrow H^n(\mathcal{A}(M))$.

Since Hattori [14] has proved that the result of Nomizu's theorem holds for compact completely solvable manifolds, then the same arguments as in Theorem 6.2 allow us to prove the following.

Theorem 6.3 [10]. *Let G be a connected completely solvable Lie group endowed with a left invariant symplectic structure ω^* and with a discrete subgroup Γ such that the space of right cosets $M = \Gamma\backslash G$ is compact. Then the natural mapping $m_c : H^k(\mathcal{A}(\mathfrak{g}^*)) \longrightarrow H^k(\mathcal{A}(M))$, defined by $m_c\{\alpha^*\} = \{\alpha\}$, is an isomorphism of cohomology groups.*

Theorems 6.2 and 6.3 have permitted us the calculation of the coeffective cohomology of the examples of compact symplectic nilmanifolds and completely solvable manifolds that appear in this article.

The above method using the long exact sequence in cohomology (5) can be applied to prove other results on the calculation of the coeffective cohomology as we show below.

First, let us recall the following theorem due to Raghunathan [23]: let G be a simply connected and solvable Lie group and let $\Gamma \subset G$ be a lattice such that Ad Γ and Ad G have the same Zariski closures $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$, then $H^k(\Gamma\backslash G) \cong H^k(\mathfrak{g}^*)$. Hence, we have:

Theorem 6.4. *Let G be a simply connected and solvable Lie group with a left invariant symplectic structure ω^*, and let $\Gamma \subset G$ be a lattice such that* Ad Γ *and* Ad G *have the same Zariski closures* $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$. *Then $H^k(\mathcal{A}(\Gamma\backslash G)) \cong H^k(\mathcal{A}(\mathfrak{g}^*))$.*

Finally, a well-known result by E. Cartan [12] states that for a connected compact Lie group we have $H^k(G) \cong H^k(\mathfrak{g}^*)$. Thus, we can obtain a similar result for the coeffective cohomology of a connected compact Lie group with a left invariant symplectic structure.

7. Deformation of symplectic structures

The purpose of this section is to study the variation of the coeffective cohomology if we perform a deformation of the symplectic structure on a compact manifold. In Section 4, we have already partially answered to this question; more precisely, we have proved that if in a compact Kähler we change the Kähler structure for another one, the coeffective cohomology remains invariant.

First of all, let us recall the following result by Moser [5,20,24].

Theorem 7.1. *If M is a compact manifold of dimension $2n$ and ω_t, for $t \in [0, 1]$, is a continuous 1-parameter family of smooth symplectic structures on M which has the property that the cohomology classes $[\omega_t]$ in $H^2(M)$ are independent of t, then for each $t \in [0, 1]$, there exists a diffeomorphism ϕ_t such $\phi_t^*(\omega_t) = \omega_0$ (that is, ϕ_t is a symplectomorphism.)*

The following definitions can be found in [19]. Two symplectic forms ω_0 and ω_1 are said to be *homotopic* if they can be joined by a smooth homotopy of non-degenerate two forms ω_t, $0 \leq t \leq 1$. If, moreover, ω_t is symplectic for every t, they are called *deformation equivalent* (or *pesudo-isotopic*), and *isotopic* if, in addition, all of them are cohomologous.

Corollary 7.1. *For a compact manifold M of dimension $2n$ with a symplectic structure ω, the coeffective cohomology does not depend on ω, but on the isotopy class of ω.*

Remark 7.1. It should be noticed that Theorem 7.1 does not hold for non-compact symplectic manifolds (see [5] for a counterexample).

But, we shall see, by constructing an example, that the coeffective cohomology is not invariant under deformations of the symplectic structure.

Example 7.1. Let R^6 be the six-dimensional compact nilmanifold mentioned in Section 4.

Consider the continuous 1-parameter family of closed 2-forms on R^6 given by:

$$\omega_t = (2-t)\alpha_{15} + \alpha_{16} + (2-t)(\alpha_{25} + \alpha_{34}) + (t-1)(\alpha_{24} + \alpha_{35}).$$

An easy computation shows that

$$\omega_t^3 = 2(3-2t)\alpha_{123456},$$

from which we deduce that ω_t defines a continuous 1-parameter family of symplectic stuctures on R^6 for $t \neq \frac{3}{2}$.

Now, from Theorem 6.2 we compute the coeffective cohomology groups for the symplectic structures ω_t:

$$\begin{aligned} H^4(\mathcal{A}(R^6, \omega_t)) = {} & \{\{\alpha_{1456}\}, \{\alpha_{1246} - (t-1)\alpha_{2345}\}, \{\alpha_{1256} - (2-t)\alpha_{2345}\}, \\ & \{\alpha_{1356} + (t-1)\alpha_{2345}\}, \{\alpha_{2346} + (2-t)\alpha_{2345}\}, e(t)\{\alpha_{1245}\}\}, \\ H^k(\mathcal{A}(R^6, \omega_t)) = {} & H^k(R^6), \quad k = 5, 6, \end{aligned} \tag{12}$$

where the function $e(t)$ takes the value 1 for $t = 1$ and the value 0 for $t \neq 1$.

Then, from (12) we obtain

$$c_4(R^6, \omega_t) = 5 + e(t). \tag{13}$$

Therefore, the symplectic structures on R^6,

$$\omega_0 = 2\alpha_{15} + \alpha_{16} + 2\alpha_{25} + 2\alpha_{34} - \alpha_{24} - \alpha_{35} \quad \text{and} \quad \omega_1 = \alpha_{15} + \alpha_{16} + \alpha_{25} + \alpha_{34}$$

are deformation equivalent but their coeffective cohomology is not the same. Indeed, from (13) we have that $c_4(R^6, \omega_0) = 5 \neq 6 = c_4(R^6, \omega_1)$, that is,

$$H^4(\mathcal{A}(R^6, \omega_0)) \ncong H^4(\mathcal{A}(R^6, \omega_1)).$$

In [19] the authors look for the existence of families of distinct symplectic structures on the same manifold; as a particular case, they are interested in symplectic structures that are deformation equivalent, but not symplectomorphic. In this direction, we have the following result.

Theorem 7.2. *The symplectic structures ω_0 and ω_1 defined on the compact nilmanifold R^6 are deformation equivalent but not symplectomorphic.*

Proof. Taking into account that the symplectic structures ω_0 and ω_1 have different coeffective cohomology, then they cannot be symplectomorphic. □

Remark 7.2. Note that in fact, we have obtained a stronger result. If we consider $\widetilde{\omega}_t = (1/(3-2t))\omega_t$, then we have a continuous 1-parameter family of symplectic structures with the same volume form. Therefore, the above example gives us an example of symplectic structures, with the same volume form, deformation equivalent by a family that preserves the volume form, but with different coeffective cohomology, and then not symplectomorphic.

Acknowledgements

This work has been partially supported through grants DGICYT (Spain), Projects PB94-0106, PB-94-0633-C02-02, and through grant UPV, Project 127.310-EC 248/96.

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